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ON THE ANALYSIS OF RESONANCES IN NONLINEAR SYSTEMS

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Perturbed resonant solutions of an essentially nonlinear real system containing two phases and a quasiconstant vector, are constructed over an infinite time interval. The first Liapunov method and well known Weierstrass theorems on implicit functions are used to derive the sufficient conditions of stability of perturbed resonant motions. The results obtained are interesting and may find application to certain problems of the theory of nonlinear oscillations.

1. Statement of the problem. We investigate, in the resonant region, a perturbed system of $(l+2)$ equations of the form

$$da / dt = \varepsilon A (\theta, a, \psi, \varepsilon)$$

$$d\psi / dt = \Omega (a) + \varepsilon \Psi (\theta, a, \psi, \varepsilon), \qquad d\theta / dt = \sigma (a) + \varepsilon N (\theta, a, \psi, \varepsilon) \qquad (1.1)$$

Here $t \in [t_0, \infty)$ is time, $\varepsilon \in [-\varepsilon_0, \varepsilon_0]$ is a small parameter, a is a quasiconstant l-dimensional vector ($|a - a_0^*| < \delta$), while ψ and θ denote scalar phases ($|\psi|, |\theta| < \infty$). We assume that the functions A, Ω, Ψ, σ and N are sufficiently smooth in all their arguments within the indicated region and are periodic in θ and ψ, the periods being equal to $2\pi / \nu$ and 2π, respectively. The degree of smoothness shall be established below. We also assume that at least one of the phases, say θ, is rotating, i.e. $\sigma (a) > 0$.

We construct solutions of (1.1) and investigate their Liapunov stability. These solutions are such that when $\varepsilon = 0$ then they have the form

$$a_0, \quad \psi_0 = \Omega (a_0) (t - t_0) + \alpha, \quad \theta_0 = \sigma (a_0)(t - t_0) + \beta$$

while for $\varepsilon \neq 0$ they do not differ appreciably from the above magnitudes for all real t (in the above expressions a_0, α and β are certain constants). In the present paper we study the resonant case, when

$$m \Omega (a_0) = n \nu \sigma (a_0) \qquad (1.2)$$

where m and n are "not very large" integers [1] and n may become equal to zero, i.e. the phases ψ may be oscillating.

System (1.1) appears in many problems of the theory of nonlinear oscillations and in particular, in the problems on forced motions in a system with one degree of freedom and little varying parameters, in certain strongly connected autonomous systems, e. a.

Similar systems were investigated in [2 and 3] using the concept of averaging over the interval $\Delta t \sim 1 / \sqrt{\varepsilon}$. A particular case of the system (1.1) ($l = 1$, $\theta \equiv \nu t$) was investigated by the author, who considered its general and particular solution [4 and 5] over the interval $t \in [t_0, \infty)$.

We also note that the analysis of the Liapunov stability of the steady resonant modes of the considered system (1.1) which is essentially nonlinear, presents great difficulties since $(l+1)$ groups of solutions correspond to the $(l+2)$-tuple null characteristic index of the unperturbed variational system. Computation of characteristic indices involves, in this case, fractional powers of ε [4–7].

2. Construction of the resonant solution. We consider the following system of $(l+1)$ equations

$$da / d\theta = \varepsilon f(\theta, a, \psi, \varepsilon), \qquad d\psi / d\theta = \omega(a) + \varepsilon F(\theta, a, \psi, \varepsilon) \tag{2.1}$$

which follows from (1.1), and where the following notation is used

$$f = A / (\sigma + \varepsilon N), \quad \omega = \Omega / \delta, \quad F = (\sigma\Psi - \Omega N) / \sigma(\sigma + \varepsilon N)$$

We assume that the functions f, ω and F satisfy the following smoothness requirements: 1) f and F are continuous in θ; (2) f and ω possess first partial derivatives in a, ψ and ε satisfying the Lipschitz conditions in these variables; (3) F satisfies the Lipschitz conditions in a, ψ and ε in the domain of definition of the system (1.1) given above. Then, employing the following substitution

$$a = a_0 + \varepsilon x, \quad \psi = (n / m)\nu(\theta - \theta_0) + \tau + \varepsilon y \qquad (a_0, \tau = \text{const}) \tag{2.2}$$

we can obtain a quasilinear system of the form

$$\frac{dx}{d\theta} = f(\theta, a_0, \psi_0, 0) + \varepsilon\left[\left(\frac{\partial f}{\partial a}\right)_0 x + \left(\frac{\partial f}{\partial \psi}\right)_0 y + \left(\frac{\partial f}{\partial \varepsilon}\right)_0 + f^*(\theta, x, y, \varepsilon)\right]$$

$$\frac{dy}{d\theta} = \left(\frac{\partial \omega}{\partial a}\right)_0 x + F(\theta, a_0, \psi_0, 0) + F^*(\theta, x, y, \varepsilon) \tag{2.3}$$

where f^* and F^* are known functions which become identically zero when $\varepsilon = 0$. A $(T = 2\pi m / \nu)$-periodic solution of this system can be constructed using the method of consecutive approximations [6–8]. The zeroth approximation to the unknown functions x and y can be found from

$$dx_0 / d\theta = f(\theta, a_0, \psi_0, 0), \quad dy_0 / d\theta = (\partial\omega / \partial a)_0 x_0 + F(\theta, a_0, \psi_0, 0)$$

and is given by

$$x_0 = \int_{\theta_0}^{\theta} f_0 d\theta_1 + c_0, \qquad y_0 = \left(\frac{\partial \omega}{\partial a}\right)_0 a_0(\theta - \theta_0) + \int_{\theta_0}^{\theta}\left(\int_{\theta_0}^{\theta_1} f_0 d\theta_2 + F_0\right) d\theta_1 + b_0$$

where c_0 and b_0 are constants of integration. Vector function x_0 will be periodic at any c_0, if the following l equations hold.

$$P(a_0, \tau) \equiv \int_{\theta_0}^{\theta_0 + T} f\left(\theta, a_0, \frac{n}{m}\nu(\theta - \theta_0) + \tau, 0\right) d\theta \equiv T\langle f_0 \rangle = 0 \tag{2.4}$$

Here and in the following expressions in angle brackets denote the averaging over the time T. Relations (2.4) and (1.2) together define the constants a_0 and τ. Let a_0^* and τ^* represent a real solution of the system. Then the function y_0 will be periodic, provided that

$$\left(\frac{\partial \omega}{\partial a}\right)_0 c_0 = -\left\langle \int_{\theta_0}^{\theta} f_0 d\theta_1 + F_0 \right\rangle \tag{2.5}$$

holds.

In this manner we obtain the following expressions for x_0 and y_0: $x_0 = x_0^* + c_0$ and $y_0 = y_0^* + b_0$ in which x_0^* and y_0^* are known T-periodic functions of θ.

First approximation is found from the equations containing terms of the order of ε and we have the following expression for the vector function x_1

$$x_1 = x_1^* + c_1 + \varepsilon \int_{\theta_1}^{\theta} \left[\left(\frac{\partial f}{\partial a}\right)_0 c_0 + \left(\frac{\partial f}{\partial \psi}\right)_0 b_0\right] d\theta_1 \qquad (c_1 = \text{const})$$

$$\left(x_1^* \equiv x_0^* + \varepsilon \int_{\theta_0}^{\theta} \left[\left(\frac{\partial f}{\partial a}\right)_0 x_0^* + \left(\frac{\partial f}{\partial \psi}\right)_0 y_0^* + \left(\frac{\partial f}{\partial \varepsilon}\right)_0\right] d\theta_1 \equiv x_0^* + \varepsilon \int_{\theta_0}^{\theta} f_1 d\theta_1\right)$$

The first approximation to x will be periodic, if the constants c_0 and b_0 satisfy the following l linear equations

$$\frac{\partial P}{\partial a_0^*} c_0 + \frac{\partial P}{\partial \tau^*} b_0 = -\langle f_1 \rangle$$

which, together with the linear equation (2.5) form a system defining c_0 and b_0. The determinant

$$\Delta = \partial(\omega_0, P) / \partial(a_0^*, \tau^*) \qquad (2.6)$$

of this system will, in the following, be assumed different from zero, i.e. the system (1.2), (2.4) admits a simple, real root (a_0^*, τ^*).

Thus we have fully defined the periodic functions x_0 and y_0. Function y_1 is obtained similarly

$$y_1 = \int_{\theta_0}^{\theta} (F_1 - \langle F_1 \rangle) d\theta_1 + b_1 \equiv y_1^* + b_1 \qquad (b_1 = \text{const})$$

$$(F_1(\theta, \varepsilon) \equiv F_0 + F^*(\theta, x_0, y_0, \varepsilon), \ (\partial\omega / \partial a)_0 c_1 = -\langle F_1 \rangle)$$

and higher approximations to x and y can be obtained from

$$\frac{dx_k}{d\theta} = f_0 + \varepsilon \left[\left(\frac{\partial f}{\partial a}\right)_0 x_{k-1} + \left(\frac{\partial f}{\partial \psi}\right)_0 y_{k-1} + \left(\frac{\partial f}{\partial \varepsilon}\right)_0 + f^*(\theta, x_{l-1}, y_{k-1}, \varepsilon)\right]$$

$$\frac{dy_k}{d\theta} = \left(\frac{\partial \omega}{\partial a}\right)_0 x_k + F_0 + F^*(\theta, x_{k-1}, y_{k-1}, \varepsilon) \qquad (2.7)$$

into which we insert, consecutively, (x_1, y_1) etc. It can be shown by induction, that the above method will yield any approximation to the periodic solution of (2.3). Indeed, let

$$x_{p-1} = x_{p-1}^* + c_{p-1}, \quad y_{p-1} = y_{p-1}^* + b_{p-1} \quad (p \geqslant 1)$$

be known periodic functions of θ. Then the right hand side of the vector equation for x_p will also be known, and this implies that $x_p = x_p^* + c_p$ where c_p is a constant vector. Applying the same process to y_p, we obtain

$$y_p = y_p^* + b_p, \qquad (\partial\omega / \partial a)_0 c_p = -\langle (\partial\omega / \partial a)_0 x_p^* + F_0 + F_{p-1}^* \rangle$$

Condition of periodicity of the vector function x_{p+1} yields the following system for the constants c_p and b_p:

$$\frac{\partial P}{\partial a_0^*} c_p + \frac{\partial P}{\partial \tau^*} b_p + \langle f^*(\theta, x_p^* + c_p, y_p^* + b_p, \varepsilon) \rangle = -\left\langle \left(\frac{\partial f}{\partial a}\right)_0 x_p^* + \left(\frac{\partial f}{\partial \psi}\right)_0 y_p^* + \left(\frac{\partial f}{\partial \varepsilon}\right)_0 \right\rangle$$

Using well known theorems on implicit functions we can find, for sufficiently small ε, a unique solution for the above system. The required roots can be found using the method of consecutive approximations

$$(\partial\omega / \partial a)_0 c_{p,i} = -\langle (\partial\omega / \partial a)_0 x_p^* + F_0 + F_{p-1}^* \rangle$$

$$\frac{\partial P}{\partial a_0^*} c_{p,i} + \frac{\partial P}{\partial \tau^*} b_{p,i} + \langle f^*(\theta, x_p^* + c_{p,i-1}, y_p^* + b_{p,i-1}, \varepsilon) \rangle =$$

$$= -\left\langle \left(\frac{\partial f}{\partial a}\right)_0 x_p^* + \left(\frac{\partial f}{\partial \psi}\right)_0 y_p^* + \left(\frac{\partial f}{\partial \varepsilon}\right)_0 \right\rangle$$

$$(c_{p,0} = c_0, \ b_{p,0} = b_0)$$

Thus we have constructed a periodic solution of (2.3) which is unique for fixed a_0^* and

τ^*, together with a resonant solution of the form (2.2), to the system (2.1). A proof of the the above scheme of consecutive approximations is given in [7 and 8].

To construct a solution to the initial system (1.1), we shall insert the functions (2.2) into its last equation. This will yield the following equation with separable variables:

$$d\theta / dt = \sigma (a_0^* + \varepsilon x (\theta, \varepsilon)) + \varepsilon N (\theta, a_0^* + \varepsilon x (\theta, \varepsilon))$$

$$(n / m) \nu (\theta - \theta_0) + \tau^* + \varepsilon y (\theta, \varepsilon), \varepsilon) \equiv 1 / M (\theta, \varepsilon) \geqslant \mu > 0$$

where M is a known bounded T-periodic function of θ which, in accordance with (1.2), becomes equal to $n\nu / m\Omega$ when $\varepsilon = 0$.

From this equation we obtain

$$\theta = \frac{1}{\langle M \rangle} (t - t_0) + \gamma + \frac{1}{\langle M \rangle} \int_{\theta_0}^{\theta} (\langle M \rangle - M) \, d\theta_1$$

and applying the scheme of consecutive approximations $(j \geqslant 1)$,

$$\theta_j = \frac{1}{\langle M \rangle} (t - t_0) + \gamma + \frac{1}{\langle M \rangle} \int_{\theta_0}^{\theta_{j-1}} (\langle M \rangle - M) \, d\theta$$

$$\theta_0 = (1 / \langle M \rangle) (t - t_0) + \gamma \equiv \varphi \qquad (\gamma = \text{const} \in (-\infty, \infty))$$

we arrive at the general solution of the form

$$\theta = \varphi + \varepsilon w (\varphi, \varepsilon) \tag{2.8}$$

Since the function w is T-periodic in φ, the quantity θ has a constant increment T over any complete interval of time, the increment being equal to $\Delta t = T \langle M \rangle \equiv \Pi$. The remaining unknowns a and ψ are obtained by inserting (2.7) into (2.2), which gives

$$a = a_0^* + \varepsilon x (\varphi + \varepsilon w (\varphi, \varepsilon), \varepsilon) \tag{2.9}$$

$$\psi = (n / m) \nu [(1 / \langle M \rangle)(t - t_0) + \gamma - \theta_0] + \tau^* + \varepsilon y (\varphi + \varepsilon w (\varphi, \varepsilon), \varepsilon)$$

Function ψ receives the constant increment equal to $2\pi n$ over any $\Delta t = \Pi$, and y and a are Π-periodic.

3. Investigation of Liapunov stability. We perform, in (2.1), the following substitution $\quad a = a (\theta, \varepsilon) + U, \ \psi = \psi (\theta, \varepsilon) + V$

in which $a (\theta, \varepsilon)$ and $\psi (\theta, \varepsilon)$ represent the resonant solution (2.2) of the system (2.1), constructed in the previous Section. Thus the problem reduces to obtaining such values of λ, for which the system

$$\frac{du}{d\theta} = \left(\varepsilon \frac{\partial f}{\partial a} - \lambda I\right) u + \varepsilon \frac{\partial f}{\partial \psi} v, \qquad \frac{dv}{d\theta} = \left(\frac{\partial \omega}{\partial a} + \varepsilon \frac{\partial F}{\partial a}\right) u + \left(\varepsilon \frac{\partial F}{\partial \psi} - \lambda\right) v \tag{3.1}$$

admits a T-periodic solution. Using the method of consecutive approximations we can show that some of the characteristic indices λ are of the order of a fractional power of ε, and that when $(\partial P / \partial \tau^*)(\partial \omega / \partial a)_0 \neq 0$, then two of them are of the order of $\delta = \sqrt{\varepsilon}$, and the remaining ones, of the order of ε. Consequently, the periodic solution of (3.1) and the required indices are of the form

$$u (\theta, \varepsilon) = u_0 (\theta) + \delta u_1 (\theta) + \delta^2 u_2 (\theta) + \delta^3 u_3 (\theta, \delta)$$

$$v (\theta, \varepsilon) = v_0 (\theta) + \delta v_1 (\theta) + \delta^2 v_2 (\theta, \delta), \ \lambda = \delta \lambda_1 + \delta^2 \lambda_2 (\delta)$$

Using consecutive approximations analogous to (2.7) we can show that λ_1 satisfies the following $(l + 1)$-th order equation:

$$(-\lambda_1)^{l-1} [\lambda_1^2 - (\partial P / \partial \tau^*) (\partial \omega / \partial a)_0 / T] = 0$$

and this implies that $(l - 1)$ values of λ_1 become zero, and remaining two are, respectively, $\pm [(\partial P / \partial \tau^*)(\partial \omega / \partial a)_0 / T]^{1/2}$. Further, we can establish that $(\partial P / \partial \tau^*)(\partial \omega / \partial a)_0 < 0$ is the necessary condition of stability for the constructed resonant solution.

Conditions of periodicity of the functions $v_2(\theta, 0)$ and $u_3(\theta, 0)$ imply that the following $(l-1)$-th order equation is the defining equation for the quantities $\lambda_2(0)$ corresponding to the values $\lambda_1 = 0$ (3.2)

$$D(\lambda_2(0)) \equiv \begin{vmatrix} (\partial\omega/\partial a_1)_0 & \cdots & (\partial\omega/\partial a_l)_0 & 0 \\ \partial P_1/\partial a_{10}^* - \lambda_2(0)\, T & \cdots & \partial P_1/\partial a_{l0}^* & \partial P_1/\partial\tau^* \\ \cdots & \cdots & \cdots & \cdots \\ \partial P_l/\partial a_{10}^* & \cdots & \partial P_l/\partial a_{l0}^* - \lambda_2(0)\, T & \partial P_l/\partial\tau^* \end{vmatrix} = 0$$

It should be noted that none of the quantities λ_2 corresponding to the values $\lambda_1 = 0$ are equal to zero, since by (2.6), $D(0) = \Delta \neq 0$. Thus the considered solution can be stable, provided that the real parts of all roots are nonpositive. To complete the discussion of the sufficient conditions, we must compute the reamining two values of λ_2 using the conditions of periodicity of the functions v_2 and u_3 given above. The Kronecker-Capelli theorem yields the following expressions for these two values

$$2\lambda_2(0)(-\lambda_1)^l - \frac{1}{\lambda^l}\begin{vmatrix} -\lambda_1^2 & 0 & \cdots & 0 & d_1 \\ 0 & -\lambda_1^2 & \cdots & 0 & d_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & -\lambda_1^2 & d_l \\ (\partial\omega/\partial a_1)_0 & (\partial\omega/\partial a_2)_0 & \cdots & (\partial\omega/\partial a_l)_0 & d_{l+1} \end{vmatrix} = 0$$

where $(j = 1, \ldots, l)$

$$d_j = \frac{1}{T^2}\sum_{k=1}^{l}\frac{\partial P_j}{\partial a_{k,0}^*}\frac{\partial P_k}{\partial \tau^*} - \lambda_1^2 T\left\langle\int_{\theta_0}^{\theta}\left[\left(\frac{\partial f_j}{\partial \psi}\right)_0 - \frac{1}{T}\frac{\partial P_j}{\partial \tau^*}\right]d\theta_1\right\rangle$$

$$d_{l+1} = \left\langle\left(\frac{\partial F}{\partial \psi}\right)_0 + \left(\frac{\partial \omega}{\partial a}\right)_0\int_{\theta_0}^{\theta}\left[\left(\frac{\partial f}{\partial \psi}\right)_0 - \frac{1}{T}\frac{\partial P}{\partial \tau^*}\right]d\theta_1\right\rangle$$

From this it follows that all $(l+1)$ characteristic indices of the first approximation system have the form

$$\lambda_{(1,\ldots,l-1)} = \varepsilon\lambda^*_{(1,\ldots,l-1)} + O(\varepsilon^{3/2})$$

$$\lambda_{l,l+1} = \pm\sqrt{\varepsilon}\left(\frac{1}{T}\left(\frac{\partial\omega}{\partial a}\right)_0\frac{\partial P}{\partial\tau^*}\right)^{1/2} + \frac{\varepsilon}{2}\left[d_{l+1} + \sum_{j=1}^{l} d_j\left(\frac{\partial\omega}{\partial a_j}\right)_0 \Big/ \sum_{j=1}^{l}\left(\frac{\partial\omega}{\partial a_j}\right)_0\frac{\partial P_j}{\partial\tau^*}\right] + O(\varepsilon^{3/2})$$

where $\lambda^*_{(1,\ldots,l-1)}$ are the roots of (3.2).

Thus we can state that the solution (2.2) is asymptotically stable for sufficiently small $\varepsilon > 0$, provided that the inequality

$$d_{l+1} + \sum_{j=1}^{l} d_j\left(\frac{\partial\omega}{\partial a_j}\right)_0 \Big/ \sum_{j=1}^{l}\left(\frac{\partial\omega}{\partial a_j}\right)_0\frac{\partial P}{\partial\tau^*} < 0$$

and the conditions shown above both hold and that the real parts of the roots of (3.2) are negative, and unstable otherwise. The case when the real parts of $\lambda_2(0)$ are zero, requires additional investigations. It should be noted that the smoothness requirements imposed on the function F are more stringent, compared with the conditions given in Section 2.

Using the Andronov-Vitte theorem [7] we can assert that the solution (2.8) and (2.9) of the system (1.1) will be Liapunov stable for sufficiently small $\varepsilon > 0$ and $t \geqslant t_0$, if all $\delta(a_0^*)\,\lambda$ have negative real parts.

When $l = 1$ [4 and 9], the conditions of stability become very simple

$$\lambda_1^2 = \frac{\omega_0'}{T} \frac{\partial P}{\partial \tau^*} < 0, \qquad \lambda_2(0) = \frac{1}{2} \left\langle \left(\frac{\partial f}{\partial a}\right)_0 + \left(\frac{\partial F}{\partial \psi}\right)_0 \right\rangle < 0$$

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THE MOVING ANGULAR VELOCITY HODOGRAPH IN HESS' SOLUTION OF THE PROBLEM OF MOTION OF A BODY WITH A FIXED POINT

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In contrast to other cases of integrable equations of motion of a body with a fixed point, determination of the components of the angular velocity of such a body in moving coordinates in Hess' solution [1] is not reducible to quadratures; it reduces to a Riccati differential equation, which complicates investigation considerably.

The case of integrability pointed out by Hess has been investigated by many authors, largely by analytical methods [2–7]. A geometric interpretation of the motion of a body in this solution was given by Zhukovskii [8], who used an intermediate moving coordinate system.

The present paper contains a direct interpretation (i. e. one which does not involve intermediate coordinate systems) based on Kharlamov's dynamic [9 and 10] and kinematic [11 and 12] equations.